

Dynamic response of a viscously damped cantilever with a viscous end condition

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Received 17 January 2005; received in revised form 19 April 2006; accepted 24 April 2006

Available online 25 July 2006

Abstract

This paper deals with the dynamical analysis of a cantilevered Bernoulli–Euler beam subjected to distributed external viscous damping in-span and with a viscous end condition by a single damper. In order to evaluate the vibration characteristics of the system, a procedure is presented where overdamped and underdamped “modes” are investigated simultaneously, via the dynamic stiffness matrix of the system. Further, the orthogonality conditions, which allow the decoupling of the equations of motion in terms of principal coordinates, are derived. Then, the complex frequency response function is obtained through a formula which was established for the receptance matrix previously. Finally, the dynamic impulse response function of the system is evaluated in the time domain. Comparison with the numerical results obtained via a boundary value formulation justifies the approaches used here.

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1. Introduction

The dynamic response of a cantilevered beam subjected to distributed external viscous damping in-span and with a viscous boundary condition by a single damper is important for structural engineers. Designers often use dampers in order to reduce force transmissibility and displacements of various structures. The need for reduced force transmissibility and displacements is evident since lower force levels permit simpler and lighter structural designs. Thus, dampers are currently critical in many systems, such as buildings, cars and airplanes. The dynamics of structures with fixed and free boundary conditions has been studied previously [1–3]. These analyses form the basis for many classical beam problems. Their self-adjoint operators are separated using mutually orthogonal modes to form models of structural dynamic response. However, these models admit only standing waves into the response and do not consider damping at the boundary. It is to be noted that damping is one of the most important, as well difficult and complicated problems in vibration theory of mechanical systems. Most serious problems occur in damped complex continuous systems. In systems having non-classical damping it should be taken into account that the dynamic analysis can not be carried out by the standard mode superposition method [4]. In fact, eigencharacteristics of the system are complex and the equations of motion have to be decoupled in the complex domain.

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This paper represents an approach based on the dynamic stiffness method which allows an assembly technique that enables exact free vibration analysis of either a single structural element or a combination of structural elements with different orientations. Furthermore, in contrast to finite element methods, the results using the dynamic stiffness method are not only exact, but also independent of the number of the elements used in the analysis, and thus offer much better computational efficiency [5].

Recently, a number of papers have appeared investigating different effects in beams and bars [4–8]. For a fixed-free longitudinal bar with a viscous boundary condition at its free end, a series solution was formulated and the complex eigenfunctions were evaluated in closed form [5]. The study in Ref. [4] is concerned with the application of the complex mode superposition method to the dynamic analysis of a simply supported Bernoulli–Euler beam with two rotational viscous dampers attached at the boundaries. Ref. [6] deals with the determination of the frequency response function of a cantilevered Bernoulli–Euler beam which is viscously damped by a single damper. The beam is simply supported in-span and carries a tip mass. The frequency response function is obtained through a formula that was established for the receptance matrix of discrete linear systems subjected to linear constraint equations, by considering the simple support as a linear constraint imposed on generalized coordinates. Ref. [7] is concerned with the eigencharacteristics of a continuous multi-step beam model with external damping in-span using the separation-of-variables approach. In that study, the beam considered has different physical properties in each of its parts. A cantilevered beam viscously damped by a damper with a concentrated mass at its free end has been examined in Ref. [8]. The study in Ref. [9] is concerned with the free and forced vibration analyses of a uniform cantilever beam carrying a number of spring–damper–mass systems with arbitrary magnitudes and locations were made by means of the analytical and numerical combined method.

The problem to be handled in the present paper represents to some extent a mixture of the applications mentioned above. The aim of this paper is to perform the dynamical analysis of a cantilevered Bernoulli–Euler beam subjected to distributed external viscous damping in-span and with a viscous end condition created by a single damper, by using the dynamic stiffness method. In order to evaluate the vibration characteristics of the system, a procedure is presented where overdamped and underdamped “modes” are investigated simultaneously. Then, the orthogonality conditions which allow the decoupling of the equations of motion in terms of principal coordinates are derived. Further, the complex frequency response function is obtained through a formula which was established for the receptance matrix previously [10]. Then, the dynamic impulse response function of the system is evaluated in the time domain. A further aim is to determine the amplitude distribution of the beam due to a harmonically varying vertical force acting at a given point.

Furthermore, a comparative study has been developed between the approaches used in the present study and through a boundary value problem approach of the system. Comparison with the numerical results obtained via the boundary value problem solution justifies the approach used here.

2. Theory

The problem can best be stated referring to the cantilevered beam shown diagrammatically in Fig. 1. Let it be assumed that the laterally vibrating Bernoulli–Euler beam has length L , bending rigidity EI , mass per unit length m , respectively. The beam subjected to distributed external viscous damping in-span with damping coefficient σ is assumed to be viscously damped by a damper of damping constant d at the free end. At the distance $x = \gamma L$ from the fixed end, a harmonically varying force $F(t)$ is acting on the beam.

2.1. Free vibration analysis of the system

Due to the presence of viscous damping in the system it is more appropriate to work with complex variables. It will be assumed that the lateral displacements $w(x,t)$ are the real parts of some complex quantities denoted as $z(x,t)$. Keeping in mind that, actually we are interested only in the real parts of the expressions below, the equation of motion of the beam can be written as

$$EIz^{IV}(x,t) + m\ddot{z}(x,t) + \sigma\dot{z}(x,t) = 0, \quad (1)$$

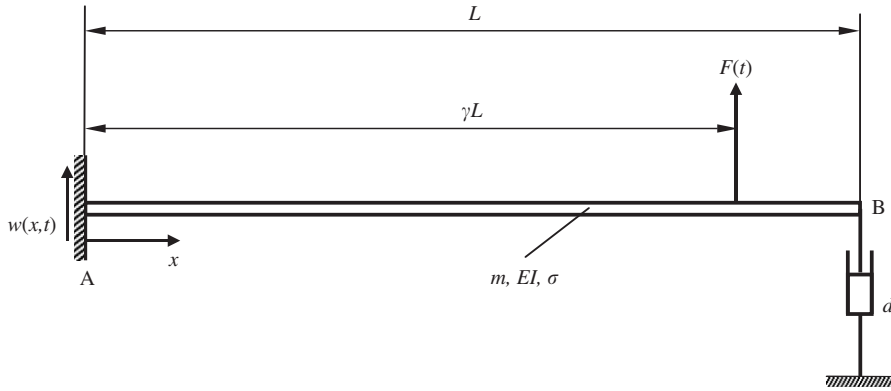


Fig. 1. Cantilevered beam subjected to external viscous damping in-span with a viscous end condition by a single damper.

where x is the axial position coordinate along the beam. Dots and primes denote partial derivatives with respect to time t and position coordinate x , respectively.

The corresponding boundary conditions are

$$z(0, t) = 0, \quad z'(0, t) = 0,$$

$$z''(L, t) = 0, \quad EIz'''(L, t) = d\dot{z}(L, t). \quad (2)$$

Let it be assumed that the solutions are of the form

$$z(x, t) = Z(x)e^{\lambda t}, \quad (3)$$

where $Z(x)$ is a complex function in general. λ represents an eigenvalue of the system which is also complex in general. Substituting Eq. (3) into Eq. (1) gives

$$Z^{IV}(x) - \beta^4 Z(x) = 0, \quad (4)$$

with the abbreviation

$$\beta^4 = -\frac{m}{EI} \left(\frac{\sigma}{m} \lambda + \lambda^2 \right). \quad (5)$$

The general solution of the differential Eq. (4) can be expressed as

$$Z(x) = \bar{C}_1 e^{\beta x} + \bar{C}_2 e^{-\beta x} + \bar{C}_3 e^{i\beta x} + \bar{C}_4 e^{-i\beta x}, \quad (6)$$

where \bar{C}_i ($i = 1, \dots, 4$) denote complex constants to be determined. In terms of the $Z(x)$, the boundary conditions in Eq. (2) can be formulated as

$$Z(0) = 0, \quad Z'(0) = 0,$$

$$Z''(L) = 0, \quad EIZ'''(L) = d\lambda Z(L). \quad (7)$$

The dynamic stiffness coefficients of a beam represent the end forces (F_A and F_B) and moments (M_A and M_B) resulting from the application of unit end displacements (Z_A and Z_B) and rotations (φ_A and φ_B) [11]. Thus, expressing the displacements and rotations at the two ends of the beam by means of the definitions given by Eq. (9) in matrix form lead to

$$\mathbf{x} = \mathbf{U}_1 \bar{\mathbf{c}}, \quad (8)$$

where

$$\mathbf{x} = [Z_A \quad Z_B \quad \varphi_A \quad \varphi_B]^T, \quad \mathbf{U}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{\beta L} & e^{-\beta L} & e^{i\beta L} & e^{-i\beta L} \\ -\beta & \beta & -i\beta & i\beta \\ -\beta e^{\beta L} & \beta e^{-\beta L} & -i\beta e^{i\beta L} & i\beta e^{-i\beta L} \end{bmatrix},$$

$$Z_A = Z(x_A), \quad Z_B = Z(x_B), \quad \varphi_A = Z'(x_A), \quad \varphi_B = Z'(x_B)$$

$$\bar{\mathbf{c}} = [\bar{C}_1 \quad \bar{C}_2 \quad \bar{C}_3 \quad \bar{C}_4]^T. \tag{9}$$

Now expressing Eq. (6) and its first three derivatives in matrix form lead to

$$\mathbf{f} = \mathbf{U}_2 \bar{\mathbf{c}}, \tag{10}$$

where

$$\mathbf{f} = [F_A \quad F_B \quad M_A \quad M_B]^T,$$

$$\mathbf{U}_2 = EI \begin{bmatrix} \beta^3 & -\beta^3 & -i\beta^3 & i\beta^3 \\ -\beta^3(1-\mu)e^{\beta L} & \beta^3(1+\mu)e^{-\beta L} & \beta^3(i+\mu)e^{i\beta L} & -\beta^3(i-\mu)e^{-i\beta L} \\ \beta^2 & \beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 e^{\beta L} & -\beta^2 e^{-\beta L} & \beta^2 e^{i\beta L} & \beta^2 e^{-i\beta L} \end{bmatrix}, \quad \mu = \frac{d\lambda}{EI\beta^3}. \tag{11}$$

It is evident that the constants \bar{C}_i can be evaluated via Eq. (8) and then substituting these into Eq. (10) leads to

$$\mathbf{f} = \mathbf{U}_2 \mathbf{U}_1^{-1} \mathbf{x}. \tag{12}$$

The central matrix product in Eq. (12) which contains the dynamic stiffness information leads, by considering the corresponding boundary conditions [11], to

$$\begin{bmatrix} F_B \\ M_B \end{bmatrix} = \mathbf{K} \begin{bmatrix} z_B \\ \varphi_B \end{bmatrix}, \tag{13}$$

where \mathbf{K} denotes the dynamic stiffness matrix. It can be shown that it takes the form

$$\mathbf{K} = \frac{EI}{1 + e^{2i\bar{\beta}} - 4e^{(1+i)\bar{\beta}} + e^{2\bar{\beta}} + e^{(2+2i)\bar{\beta}}} \begin{bmatrix} \alpha_F & \alpha \\ \alpha & \alpha_M \end{bmatrix}, \tag{14}$$

where

$$\bar{\beta} = \beta L,$$

$$\alpha_F = \beta^3((1-i) + e^{2\bar{\beta}}((-1-i) + \mu) + e^{(2+2i)\bar{\beta}}((-1+i) + \mu) + \mu - 4e^{(1+i)\bar{\beta}}\mu + e^{2i\bar{\beta}}((1+i) + \mu)),$$

$$\alpha = i\beta^2(-1 + e^{2i\bar{\beta}})(-1 + e^{2\bar{\beta}}),$$

$$\alpha_M = \beta(1+i)((-1 + ie^{2i\bar{\beta}} - ie^{2\bar{\beta}} + e^{(2+2i)\bar{\beta}})). \tag{15}$$

The eigenvalues of the system are obtained by setting to zero the determinant of the dynamic stiffness matrix:

$$\det \mathbf{K}(\bar{\beta}, \mu) = 0. \tag{16}$$

The eigenfunctions of the system can be obtained after determination of the coefficients \bar{C}_i . Substitution of Eq. (6) into the boundary conditions in Eq. (7) yields the following set of homogeneous equations for the

determination of the unknowns \bar{C}_i :

$$\begin{aligned}\bar{C}_1 + \bar{C}_2 + \bar{C}_3 + \bar{C}_4 &= 0, \\ \bar{C}_1 - \bar{C}_2 + i\bar{C}_3 - i\bar{C}_4 &= 0, \\ \bar{C}_1 e^{\bar{\beta}} + \bar{C}_2 e^{-\bar{\beta}} - \bar{C}_3 e^{i\bar{\beta}} - \bar{C}_4 e^{-i\bar{\beta}} &= 0, \\ \bar{C}_1(-1 - \mu)e^{\bar{\beta}} + \bar{C}_2(1 - \mu)e^{-\bar{\beta}} + \bar{C}_3(i - \mu)e^{i\bar{\beta}} + \bar{C}_4(-i - \mu)e^{-i\bar{\beta}} &= 0.\end{aligned}\quad (17)$$

It can be shown that the solution of this system of homogeneous equations yields the unknown complex constants, \bar{C}_i s by the expressions

$$\bar{\mathbf{c}} = \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \bar{C}_3 \\ \bar{C}_4 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{e^{\bar{\beta}}(i+e^{2i\bar{\beta}}+(1+i)e^{(1+i)\bar{\beta}})}{(-1-i)e^{i\bar{\beta}}-e^{\bar{\beta}}-ie^{(1+2i)\bar{\beta}}} \\ \frac{ie^{i\bar{\beta}}+(1+i)e^{\bar{\beta}}+e^{(2+i)\bar{\beta}}}{(1-i)e^{i\bar{\beta}}-ie^{\bar{\beta}}+e^{(1+2i)\bar{\beta}}} \\ -\frac{e^{i\bar{\beta}}(1+(1+i)e^{(1+i)\bar{\beta}}+ie^{2i\bar{\beta}})}{(1-i)e^{i\bar{\beta}}-ie^{\bar{\beta}}+e^{(1+2i)\bar{\beta}}} \end{bmatrix}.\quad (18)$$

Having obtained the coefficients \bar{C}_i , the eigenfunctions $Z(x)$ are at hand if these coefficients are substituted into Eq. (6).

The differential equations which govern the mode shapes Z_n and Z_m can be written from Eq. (4) as

$$Z_n^{\text{IV}}(x) - \beta_n^4 Z_n(x) = 0, \quad (19)$$

$$Z_m^{\text{IV}}(x) - \beta_m^4 Z_m(x) = 0. \quad (20)$$

Multiplying Eq. (19) by $Z_m(x)$ and Eq. (20) by $Z_n(x)$, and integrating between 0 and L , leads to

$$\int_0^L Z_n^{\text{IV}} Z_m dx - \beta_n^4 \int_0^L Z_n Z_m dx = 0, \quad (21)$$

$$\int_0^L Z_m^{\text{IV}} Z_n dx - \beta_m^4 \int_0^L Z_m Z_n dx = 0. \quad (22)$$

By integrating the Eqs. (21) and (22) by parts, and after some arrangements,

$$Z_m(L)Z_n'''(L) - \int_0^L Z_n'''Z_m' dx - \beta_n^4 \int_0^L Z_n Z_m dx = 0, \quad (23)$$

$$Z_n(L)Z_m'''(L) - \int_0^L Z_m'''Z_n' dx - \beta_m^4 \int_0^L Z_m Z_n dx = 0 \quad (24)$$

are obtained.

Integrating Eqs. (23) and (24) by parts once more, and performing some arrangements, yields

$$Z_m(L)Z_n'''(L) + \int_0^L Z_n''Z_m'' dx - \beta_n^4 \int_0^L Z_n Z_m dx = 0, \quad (25)$$

$$Z_n(L)Z_m'''(L) + \int_0^L Z_m''Z_n'' dx - \beta_m^4 \int_0^L Z_m Z_n dx = 0. \quad (26)$$

By applying the boundary conditions and performing some rearrangements, Eqs. (25) and (26) may be written as

$$\int_0^L Z_n'' Z_m'' dx + \frac{d\omega_n}{EI} Z_n(L)Z_m(L) - \beta_n^4 \int_0^L Z_n Z_m dx = 0, \tag{27}$$

$$\int_0^L Z_m'' Z_n'' dx + \frac{d\omega_m}{EI} Z_m(L)Z_n(L) - \beta_m^4 \int_0^L Z_m Z_n dx = 0 \tag{28}$$

with the abbreviation

$$\omega_i^2 = (\beta_i L)^4 \frac{EI}{mL^4}. \tag{29}$$

Subtraction of Eq. (27) from Eq. (28) yields

$$-\frac{d}{EI}(\omega_m - \omega_n)Z_m(L)Z_n(L) + (\beta_m^4 - \beta_n^4) \int_0^L Z_m Z_n dx = 0. \tag{30}$$

Eq. (30), for $\omega_m \neq \omega_n$, gives the first orthogonality condition as

$$(\omega_m + \omega_n) \int_0^L m Z_m Z_n dx - dZ_m(L)Z_n(L) = 0, \tag{31}$$

Subtraction of Eq. (27) multiplied by ω_m , from Eq. (28) multiplied by ω_n , leads to

$$\begin{aligned} & -\omega_m \int_0^L Z_n'' Z_m'' dx - \frac{d\omega_n \omega_m}{EI} Z_n(L)Z_m(L) + \beta_n^4 \omega_m \int_0^L Z_n Z_m dx \\ & + \omega_n \int_0^L Z_m'' Z_n'' dx + \frac{d\omega_m \omega_n}{EI} Z_m(L)Z_n(L) - \beta_m^4 \omega_n \int_0^L Z_m Z_n dx = 0. \end{aligned} \tag{32}$$

Eq. (32), for $\omega_m \neq \omega_n$ and after some more arrangements, gives the second orthogonality condition as

$$\int_0^L EI Z_m'' Z_n'' dx + \omega_n \omega_m \int_0^L Z_m Z_n dx = 0. \tag{33}$$

Since to each eigenvalue λ_n , there is its conjugate $\bar{\lambda}_n$, by taking $\lambda_m = \bar{\lambda}_n$

$$\lambda_n + \bar{\lambda}_n = 2\xi_n |\omega_n|, \quad \lambda_n \bar{\lambda}_n = |\omega_n|^2 \tag{34}$$

are obtained, where ξ_n are the modal damping factors. Thus, from Eq. (31), after some arrangements,

$$2\xi_n |\omega_n| = \frac{d|Z_n(L)|^2}{\int_0^L m |Z_n|^2 dx} + \frac{\sigma}{m} \tag{35}$$

and from the Eq. (33), after some rearrangements,

$$|\omega_n|^2 = \frac{\int_0^L EI |Z_n''|^2 dx}{\int_0^L m |Z_n|^2 dx} \tag{36}$$

are obtained.

2.2. Frequency response function of the system

The equation of motion of the beam in Fig. 1 subjected to both distributed and discrete damping at the boundary and excited by the force $F(t)$ can be written as [5,6]

$$EIz^{IV}(x, t) + m\ddot{z}(x, t) + \sigma\dot{z}(x, t) = F(t)\delta(x - \gamma L), \tag{37}$$

where $\delta(x)$ denotes the Dirac's function.

An approximate series solution of the differential equation (37) can be taken in the form

$$z(x, t) \approx \sum_{r=1}^n Z_r(x) \eta_r(t), \quad (38)$$

where the $Z_r(x)$ are the orthogonal eigenfunctions of the bare clamped–free beam, normalized with respect to the mass density, and $\eta_r(t)$ are the generalized coordinates. After substitution of expression (38) into the differential equation (37), the equation of motion becomes

$$\sum_{r=1}^n [EI Z_r^{IV}(x) \eta_r(t) + m Z_r(x) \ddot{\eta}_r(t) + \sigma Z_r(x) \dot{\eta}_r(t)] = F(t) \delta(x - \gamma L). \quad (39)$$

Then, according to Galerkin's procedure, both sides of the equation are multiplied by the s 'th eigenfunction $Z_s(x)$ and integrated over the beam length L , which, after some arrangements, yields

$$\begin{aligned} \sum_{r=1}^n \left[EI \eta_r(t) \int_0^L Z_r^{IV}(x) Z_s(x) dx + m \ddot{\eta}_r(t) \int_0^L Z_r(x) Z_s(x) dx + \sigma \dot{\eta}_r(t) \int_0^L Z_r(x) Z_s(x) dx \right] \\ = \int_0^L F(t) \delta(x - \gamma L) Z_s(x) dx \quad (s = 1, \dots, n). \end{aligned} \quad (40)$$

Integrating twice by parts and applying the boundary conditions (7), one obtains

$$\begin{aligned} \sum_{r=1}^n \left[EI \eta_r(t) \left(Z_r'''(L) Z_s(L) + \int_0^L Z_r''(x) Z_s''(x) dx \right) + m \ddot{\eta}_r(t) \int_0^L Z_r(x) Z_s(x) dx + \sigma \dot{\eta}_r(t) \int_0^L Z_r(x) Z_s(x) dx \right] \\ = \int_0^L F(t) \delta(x - \gamma L) Z_s(x) dx \quad (s = 1, \dots, n). \end{aligned} \quad (41)$$

Then, by using orthogonality conditions (31) and (33) of the eigenfunctions, after some algebraic manipulations, the system of the modal equations, i.e., the system of differential equations for the $\eta_i(t)$, is obtained as [6]

$$\ddot{\eta}_i(t) + \frac{d}{m} \sum_{j=1}^n \frac{Z_i(L) Z_j(L)}{\int_0^L Z_i(x) Z_j(x) dx} \dot{\eta}_i(t) + \frac{\sigma}{m} \dot{\eta}_i(t) + \beta_i^4 \eta_i(t) = F(t) \sum_{j=1}^n \frac{Z_j(\gamma L)}{m \int_0^L Z_i(x) Z_j(x) dx}, \quad (i = 1, \dots, n; j = 1, \dots, n). \quad (42)$$

Hence, upon making use of Eqs. (35) and (36), the system of differential equations in Eq. (42) becomes

$$\ddot{\eta}_i(t) + 2\zeta_i \omega_i^* \dot{\eta}_i(t) + \omega_i^{*2} \eta_i(t) = N_i(t) \quad (i = 1, \dots, n), \quad (43)$$

where,

$$\omega_i^{*2} = (\beta_i^* L)^4 EI / mL^4, \quad \beta_1^* L = 1.875104068712, \quad \beta_2^* L = 4.694091132974, \dots [6]$$

$$N_i(t) = F(t) \sum_{j=1}^n \frac{Z_j(\gamma L)}{m \int_0^L Z_i(x) Z_j(x) dx}. \quad (44)$$

At this point, it may be noted that $N_i(t)$ is the i th generalized force. Assuming that the forcing function is of the form,

$$F(t) = F_0 e^{i\Omega t}, \quad (45)$$

the system of differential equations in Eq. (43) can be written in matrix notation as

$$\ddot{\boldsymbol{\eta}}(t) + \mathbf{D} \dot{\boldsymbol{\eta}}(t) + \boldsymbol{\omega}^2 \boldsymbol{\eta}(t) = \mathbf{N}(t), \quad (46)$$

where

$$\begin{aligned} \boldsymbol{\eta}(t) &= [\eta_1(t) \dots \eta_n(t)]^T, \quad \boldsymbol{\omega}^2 = \mathbf{diag}(\omega_i^{*2}), \\ \mathbf{N}(t) &= F_0 \mathbf{Z}(\gamma L) e^{i\Omega t}, \quad \mathbf{D} = \sigma \mathbf{I} + d \mathbf{Z}(L) \mathbf{Z}^T(L), \\ \mathbf{Z}(x) &= [Z_1(x) \dots Z_n(x)]^T. \end{aligned} \tag{47}$$

ω_i^* ($i = 1, \dots, n$) are the eigenfrequencies of the bare cantilevered beam.

Substitution of

$$\boldsymbol{\eta}(t) = \bar{\boldsymbol{\eta}} e^{i\Omega t} \tag{48}$$

into the matrix differential equation (46) yields

$$\bar{\boldsymbol{\eta}} = \mathbf{H}(\Omega) \bar{\mathbf{N}}, \tag{49}$$

where

$$\bar{\mathbf{N}} = F_0 \mathbf{Z}(\gamma L), \quad \bar{\boldsymbol{\eta}}(t) = [\bar{\eta}_1(t) \dots \bar{\eta}_n(t)]^T \tag{50}$$

Then, the receptance matrix is obtained in the form

$$\mathbf{H}(\Omega) = \{-\Omega^2 \mathbf{I} + i\Omega \mathbf{D} + \boldsymbol{\omega}^2\}^{-1}. \tag{51}$$

Considering Eq. (47), it can be arranged as

$$\mathbf{H}(\Omega) = (\mathbf{K} + \mathbf{u} \mathbf{v}^T)^{-1}, \tag{52}$$

where

$$\mathbf{K} = \boldsymbol{\omega}^2 + \Omega(i\sigma - \Omega) \mathbf{I}, \quad \mathbf{u} = i\Omega d \mathbf{Z}(L), \quad \mathbf{v}^T = \mathbf{Z}^T(L). \tag{53}$$

Using the Sherman-Morrison formula

$$(\mathbf{K} + \mathbf{u} \mathbf{v}^T)^{-1} = \mathbf{K}^{-1} - \mathbf{K}^{-1} \mathbf{u} (1 + \mathbf{v}^T \mathbf{K}^{-1} \mathbf{u})^{-1} \mathbf{v}^T \mathbf{K}^{-1} \tag{54}$$

which gives the inverse of the sum of a regular matrix and a dyadic product [6], the receptance matrix can be written as follows

$$\mathbf{H}(\Omega) = \mathbf{K}^{-1} - \frac{\mathbf{K}^{-1} i\Omega d \mathbf{Z}(L) \mathbf{Z}^T(L) \mathbf{K}^{-1}}{1 + \mathbf{Z}^T(L) \mathbf{K}^{-1} i\Omega d \mathbf{Z}(L)}, \tag{55}$$

noting that

$$\mathbf{K}^{-1} = \mathbf{diag} \left(\frac{1}{\omega_i^2 + i\sigma\Omega - \Omega^2} \right). \tag{56}$$

The receptance matrix can be expressed as

$$\begin{aligned} \mathbf{H}(\Omega) &= \mathbf{diag} \left(\frac{1}{\omega_i^2 + i\sigma\Omega - \Omega^2} \right) \\ &\left[\mathbf{I} - \frac{i \frac{d}{mL} \Omega \mathbf{a}(L) \mathbf{a}^T(L) \mathbf{diag} \left(\frac{1}{\omega_i^2 + i\sigma\Omega - \Omega^2} \right)}{1 + i \frac{d}{mL} \Omega \mathbf{a}^T(L) \mathbf{diag} \left(\frac{1}{\omega_i^2 + i\sigma\Omega - \Omega^2} \right) \mathbf{a}(L)} \right], \end{aligned} \tag{57}$$

where the following definitions are introduced:

$$\mathbf{Z}^T(x) = \frac{1}{\sqrt{mL}} \mathbf{a}^T(x) = \frac{1}{\sqrt{mL}} [a_1(x) \dots a_n(x)],$$

$$a_i(x) = \cosh \bar{\beta}_i \frac{x}{L} - \cos \bar{\beta}_i \frac{x}{L} - \bar{\eta}_i^* \left(\sinh \bar{\beta}_i \frac{x}{L} - \sin \bar{\beta}_i \frac{x}{L} \right), \quad \bar{\eta}_i^* = \frac{\cosh \bar{\beta}_i + \cos \bar{\beta}_i}{\sinh \bar{\beta}_i + \sin \bar{\beta}_i}. \tag{58}$$

Finally, after some rearrangements, the receptance matrix can be expressed as

$$\mathbf{H}(\Omega) = \mathbf{diag} \left(\frac{1}{\omega_i^2 + i\sigma\Omega - \Omega^2} \right) \left[\mathbf{I} - \frac{\mathbf{a}(L)\mathbf{a}^T(L)\mathbf{diag} \left(\frac{1}{\omega_i^2 + i\sigma\Omega - \Omega^2} \right)}{\frac{mL}{i d\Omega} + \mathbf{a}^T(L)\mathbf{diag} \left(\frac{1}{\omega_i^2 + i\sigma\Omega - \Omega^2} \right)\mathbf{a}(L)} \right] \quad (59)$$

\mathbf{I} being the $n \times n$ unit matrix.

Therefore, the displacements of the beam can be written as

$$z(x, t) = \bar{Z}(x)e^{i\Omega t}, \quad (60)$$

where

$$\bar{Z}(x) = \sum_{r=1}^n Z_r(x)\bar{\eta}_r. \quad (61)$$

It is easy to show that the above expression can be reformulated as

$$\bar{Z}(x) = (\mathbf{Z}^T(x)\mathbf{H}(\Omega)\mathbf{Z}(\gamma L))F_0. \quad (62)$$

As previously mentioned, the real part of $z(x, t)$ represents the physical displacements. In the case of $F_0 = 1$, the right-hand side of Eq. (62) represents the frequency response function of the beam in Fig. 1.

2.3. Modal impulse response function of the system

In this section, the complex mode superposition method for the dynamical analysis of the system in time domain will be presented. At first, the modal impulse response functions will be derived. Then, these functions will be used to evaluate the response to a general input.

Due to the impulsive nature of the excitation, the principal coordinates take the form [4]

$$\eta_r(t) = C_r e^{i\omega_r t}, \quad (63)$$

which imply

$$\dot{\eta}_r(t) = i\omega_r \eta_r(t), \quad \ddot{\eta}_r(t) = i\omega_r \dot{\eta}_r(t) = -\omega_r^2 \eta_r(t). \quad (64)$$

Through application of the boundary conditions (2) onto the differential equations (41), one obtains

$$\sum_{r=1}^n \left[EI\eta_r(t) \left(\frac{id\omega_r}{EI} Z_r(L)Z_s(L) + \int_0^L Z_r''(x)Z_s''(x) dx \right) + m\ddot{\eta}_r(t) \int_0^L Z_r(x)Z_s(x) dx + \sigma\dot{\eta}_r(t) \int_0^L Z_r(x)Z_s(x) dx \right] = \int_0^L F(t)\delta(x - \gamma L)Z_s(x) dx, \quad (s = 1, \dots, n). \quad (65)$$

Then, by using orthogonality conditions (31) and (33) of the eigenfunctions and applying Eqs. (40) between the principal coordinates and their derivatives (64), after some algebraic manipulations, the following equations are found

$$\sum_{r=1}^n \left[\ddot{\eta}_r(t) 2 \int_0^L mZ_r(x)Z_s(x) dx + \dot{\eta}_r(t) \left(dZ_r(L)Z_s(L) + \sigma \int_0^L Z_r(x)Z_s(x) dx \right) \right] = F(t)Z_s(\gamma L) \quad (s = 1, \dots, n). \quad (66)$$

It may be noted that, by applying the orthogonality conditions (31) and (33), the previous expressions vanish for $r \neq s$, this making it possible to decouple the equations of motion, which take the form

$$\ddot{\eta}_r(t) 2 \int_0^L mZ_r^2(x) dx + \dot{\eta}_r(t) \left(dZ_r^2(L) + \sigma \int_0^L Z_r^2(x) dx \right) = F(t)Z_r(\gamma L) \quad (r = 1, \dots, n). \quad (67)$$

At this stage, it is assumed that the forcing function is of the impulsive form:

$$F(t) = \bar{I}\delta(t), \tag{68}$$

where \bar{I} is the unit impulse and $\delta(t)$ is the Dirac's delta function.

Integrating of Eqs. (67) in the time interval $(0^-, 0^+)$ leads to

$$\dot{\eta}_r(0^+)2 \int_0^L mZ_r^2(x) dx + \eta_r(0^+) \left(dZ_r^2(L) + \sigma \int_0^L Z_r^2(x) dx \right) = \bar{I}Z_r(\gamma L) \quad (r = 1, \dots, n) \tag{69}$$

from which the constants C_r in Eq. (63) are obtained as

$$C_r = \frac{\bar{I}Z_r(\gamma L)}{2i\omega_r \int_0^L mZ_r^2(x) dx + dZ_r^2(L) + \sigma \int_0^L Z_r^2(x) dx} = \bar{I}B_r \quad (r = 1, \dots, n), \tag{70}$$

Then, the complex modal impulse response functions can be written as

$$h_r^c(x, t) = B_r Z_r(x) e^{i\omega_r t} \quad (r = 1, \dots, n). \tag{71}$$

The expression of the complex modal impulse response function is given in Appendix A due to space limitations.

However, the dynamical response can be evaluated more conveniently by using real variables. This can be performed by summing each modal contribution and its conjugate one. Thus,

$$h_r^{re}(x, t) = 2Re [B_r Z_r(x) e^{i\omega_r t}] \quad (r = 1, \dots, n). \tag{72}$$

Additionally, the impulse response function and the frequency response function are related to each other. This relationship, by using Eq. (62), may be written as

$$h(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbf{Z}^T(x) \mathbf{H}(\omega) \mathbf{Z}(\gamma L) e^{i\omega t} d\omega. \tag{73}$$

By using the impulse response function (73), the dynamical response to a general force $F(t)$ may be obtained as

$$z(x, t) = \sum_{r=1}^{\infty} \int_0^t F(\tau) h_r^{re}(x, t - \tau) d\tau, \tag{74}$$

where the loading history has been divided into a sequence of impulses, as it is usual in the convolution integral method of dynamical analysis.

When the forcing function is harmonic, the dynamical response may still be evaluated by means of Eq. (74).

2.4. Solution through the boundary value problem formulation

In order to prove the validity of the above expressions, it is reasonable to compare the results with the results of a boundary value problem formulation.

It will be assumed that the bending displacements $w_i(x, t)$, ($i = 1, 2$) of the two parts of the beam are the real parts of some complex quantities denoted as $z_i(x, t)$. The bending vibrations of the two beam portions left and right to the external force $F(t)$ shown in Fig. 1 are governed by the partial differential equations

$$EIz_i^{iv}(x, t) + m\ddot{z}_i(x, t) + \sigma\dot{z}_i(x, t) = 0 \quad (i = 1, 2) \tag{75}$$

which are formally the same as in Eq. (1). The corresponding boundary and matching conditions are as follows:

$$\begin{aligned} z_1(0, t) = 0, \quad z_1'(0, t) = 0, \\ z_1(\gamma L, t) = z_2(\gamma L, t), \quad z_1'(\gamma L, t) = z_2'(\gamma L, t), \quad z_1''(\gamma L, t) = z_2''(\gamma L, t), \\ EIz_1'''(\gamma L, t) - EIz_2'''(\gamma L, t) + F_0 e^{i\Omega t} = 0, \quad z_2''(L, t) = 0, \quad EIz_2'''(L, t) - d\dot{z}_2(L, t) = 0. \end{aligned} \tag{76}$$

If harmonic solutions of the form

$$z_i(x, t) = Z_i(x)e^{i\Omega t} \tag{77}$$

are substituted into Eq. (75), the previous ordinary differential equations in Eq. (4) are obtained for the amplitude functions $Z_i(x)$:

$$Z_i^w(x) - \beta^4 Z_i(x) = 0 \quad (i = 1, 2), \tag{78}$$

where β^4 is now defined as

$$\beta^4 = -\frac{m\Omega^2 + \sigma\Omega}{EI}. \tag{79}$$

The general solutions of the differential equations (78) are similar in form to Eq. (6)

$$\begin{aligned} Z_1(x) &= c_1 e^{\beta x} + c_2 e^{-\beta x} + c_3 e^{i\beta x} + c_4 e^{-i\beta x}, \\ Z_2(x) &= c_5 e^{\beta x} + c_6 e^{-\beta x} + c_7 e^{i\beta x} + c_8 e^{-i\beta x}. \end{aligned} \tag{80}$$

The corresponding boundary and matching conditions read now, as

$$\begin{aligned} Z_1(0) &= 0, \quad Z_1'(0) = 0, \\ Z_1(\gamma L) &= Z_2(\gamma L), \quad Z_1'(\gamma L) = Z_2'(\gamma L), \quad Z_1''(\gamma L) = Z_2''(\gamma L), \\ Z_1'''(\gamma L) - Z_2'''(\gamma L) + \frac{F_0}{EI} &= 0, \quad Z_2(L) = 0, \quad Z_2'''(L) - \frac{id\Omega}{EI} Z_2(L) = 0, \end{aligned} \tag{81}$$

where c_1 to c_8 are unknown integration constants to be determined, which can be complex in general.

The substitution of expressions (80) into Eq. (81) yields the following set of eight inhomogeneous equations for the determination of the eight unknowns c_1 to c_8 :

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & i & -i & 0 & 0 & 0 & 0 \\ e^{\bar{\beta}\gamma} & e^{-\bar{\beta}\gamma} & e^{i\bar{\beta}\gamma} & e^{-i\bar{\beta}\gamma} & -e^{\bar{\beta}\gamma} & -e^{-\bar{\beta}\gamma} & -e^{i\bar{\beta}\gamma} & -e^{-i\bar{\beta}\gamma} \\ e^{\bar{\beta}\gamma} & -e^{-\bar{\beta}\gamma} & ie^{i\bar{\beta}\gamma} & -ie^{-i\bar{\beta}\gamma} & -e^{\bar{\beta}\gamma} & e^{-\bar{\beta}\gamma} & -ie^{i\bar{\beta}\gamma} & ie^{-i\bar{\beta}\gamma} \\ e^{\bar{\beta}\gamma} & e^{-\bar{\beta}\gamma} & -e^{i\bar{\beta}\gamma} & -e^{-i\bar{\beta}\gamma} & -e^{\bar{\beta}\gamma} & -e^{-\bar{\beta}\gamma} & e^{i\bar{\beta}\gamma} & e^{-i\bar{\beta}\gamma} \\ e^{\bar{\beta}\gamma} & -e^{-\bar{\beta}\gamma} & -ie^{i\bar{\beta}\gamma} & ie^{-i\bar{\beta}\gamma} & -e^{\bar{\beta}\gamma} & e^{-\bar{\beta}\gamma} & ie^{i\bar{\beta}\gamma} & -ie^{-i\bar{\beta}\gamma} \\ 0 & 0 & 0 & 0 & e^{\bar{\beta}} & e^{-\bar{\beta}} & -e^{i\bar{\beta}} & -e^{-i\bar{\beta}} \\ 0 & 0 & 0 & 0 & (1 - \frac{id\Omega}{\beta^3 EI})e^{\bar{\beta}} & -(1 + \frac{id\Omega}{\beta^3 EI})e^{-\bar{\beta}} & -(i + \frac{id\Omega}{\beta^3 EI})e^{i\bar{\beta}} & (i - \frac{id\Omega}{\beta^3 EI})e^{-i\bar{\beta}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{F_0}{EI} \\ 0 \\ 0 \end{bmatrix}, \tag{82}$$

where $\bar{\beta} = \beta L$.

For the free vibration analysis, let the 8×8 matrix of coefficients in Eqs. (82) be denoted by **A**. For a non-trivial solution, the determinant of this matrix should be zero

$$\det \mathbf{A} = 0, \tag{83}$$

from which eigenvalues of the system can be obtained, which are complex numbers in general. If the eigenvalues of the system obtained from Eq. (83) are substituted into the coefficient matrix **A** in Eq. (82), the unknowns c_1 – c_8 can be determined. Hence, $Z_i(x)$, $i = 1, 2$ in Eqs. (80) are obtained.

Thus, the bending displacements of the beam portions $w_i(x, t)$ are determined as

$$w_i(x, t) = Re[z_i(x, t)]. \tag{84}$$

$w_i(x, t)$, $i = 1, 2$ determine the bending displacement distribution over the length of the cantilever beam subjected to external viscous damping in-span and with a viscous end condition by a single damper, when it vibrates at an eigenvalue β . Absolute values of $Z_i(x)$ represent the amplitude distribution over the i th step of the beam.

Table 1
Eigenvalue parameters β of the system for three different viscous damping coefficients d at the free end and for three different external viscous damping coefficients σ

	$\sigma = 10 \text{ kg/ms}$			$\sigma = 100 \text{ kg/ms}$		
	From Eq. (15)	From Eq. (83)	From Eq. (15)	From Eq. (83)	From Eq. (15)	From Eq. (83)
	$d = 0$	$\pm 25.84028i$ $\pm 161.93829i$ $\pm 453.43191i$ $\pm 888.54556i$ $\pm 1468.82948i$	$\pm 25.84028i$ $\pm 161.93829i$ $\pm 453.43191i$ $\pm 888.54556i$ $\pm 1468.82948i$	$-7.40741 \pm 24.75582i$ $-7.40741 \pm 161.76879i$ $-7.40741 \pm 453.37140i$ $-7.40741 \pm 888.51469i$ $-7.40741 \pm 1468.81080i$	$-7.40741 \pm 24.75582i$ $-7.40741 \pm 161.76879i$ $-7.40741 \pm 453.37140i$ $-7.40741 \pm 888.51469i$ $-7.40741 \pm 1468.81080i$	-4.65327 $-74.07407 \pm 144.00362i$ $-74.07407 \pm 447.34050i$ $-74.07407 \pm 885.45257i$ $-74.07407 \pm 1466.96049i$
$d = 10 \text{ kg/s}$	$-31.28305 \pm 15.00780i$ $-26.97632 \pm 149.34470i$ $-28.67702 \pm 446.16697i$ $-29.10793 \pm 883.31688i$ $-29.29913 \pm 1464.73711i$	$-31.28305 \pm 15.00780i$ $-26.97632 \pm 149.34470i$ $-28.67702 \pm 446.16697i$ $-29.10793 \pm 883.31688i$ $-29.29913 \pm 1464.73711i$	-10.18582 $-33.29237 \pm 148.06339i$ $-35.86876 \pm 445.65466i$ $-36.43492 \pm 883.04899i$ $-21.93038 \pm 1464.86407i$	-10.18582 $-33.29237 \pm 148.06339i$ $-35.86876 \pm 445.65466i$ $-36.43492 \pm 883.04899i$ $21.93038 \pm 1464.86407i$	-3.26233 $-92.03512 \pm 122.11581i$ $-100.63970 \pm 435.72825i$ $-102.38499 \pm 877.88844i$ $-102.98895 \pm 1461.42522i$	-3.26233 $-92.03512 \pm 122.11581i$ $-100.63970 \pm 435.72825i$ $-102.38499 \pm 877.88844i$ $-102.98895 \pm 1461.42522i$
$d = 100 \text{ kg/s}$	-1.09566 $-5.32866 \pm 113.62035i$ $-18.16303 \pm 369.79653i$ $-37.13137 \pm 774.62234i$ $-61.55170 \pm 1329.90543i$	-1.09566 $-5.32866 \pm 113.62035i$ $-18.16303 \pm 369.79653i$ $-37.13137 \pm 774.62234i$ $-61.55170 \pm 1329.90543i$	-1.07033 $-12.67971 \pm 113.72146i$ $-25.44765 \pm 370.07296i$ $-44.35743 \pm 774.91183i$ $-68.72289 \pm 1330.16913i$	-1.07033 $-12.67971 \pm 113.72146i$ $-25.44765 \pm 370.07296i$ $-44.35743 \pm 774.91183i$ $-68.72289 \pm 1330.16913i$	-0.88520 $-78.88160 \pm 90.17755i$ $-91.06931 \pm 365.65266i$ $-109.45781 \pm 774.18387i$ $-133.33482 \pm 1330.58117i$	-0.88520 $-78.88160 \pm 90.17755i$ $-91.06931 \pm 365.65266i$ $-109.45781 \pm 774.18387i$ $-133.33482 \pm 1330.58117i$

Finally, in the following, the frequency response function of the beam is represented. From Eq. (82) the unknowns c_1 – c_8 can be determined as

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{bmatrix} = -\frac{F_0}{EI\kappa} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \\ \chi_6 \\ \chi_7 \\ \chi_8 \end{bmatrix}, \tag{85}$$

where κ and χ_1 – χ_8 are given in Appendix B.

Having obtained $Z_i(x)$ ($i = 1, 2$), it is possible to determine the steady-state amplitude at any point x of the beam, due to the harmonic force at a point $x = \gamma L$. Noting that according to Eq. (77) the real part of $Z_i(x)e^{i\Omega t}$ represents the physical displacements, absolute values of $Z_i(x)$ represent the amplitude distribution along the cantilevered beam subjected to the harmonically varying vertical force at $x = \gamma L$. In the case of $F_0 = 1$, $|Z_i(x)|$ represents the frequency response function of the beam in Fig. 1.

Then the impulse response function of the system, by using Eqs. (80), may be written as

$$h_r(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |Z_r(x)| e^{i\omega t} d\omega, \quad r = 1, 2. \tag{86}$$

3. Numerical applications

This section is devoted to the numerical evaluations of the formulae established in the preceding sections. In the examples, the length $L = 1$ (m), the bending rigidity $EI = (7 \times 10^{10}) \frac{0.05 \times 0.05^3}{12}$ (Nm²), mass per unit length $m = 0.675$ (kg/m) and the non-dimensional position of the harmonically varying vertical force $\gamma = 1.0$ are chosen, respectively.

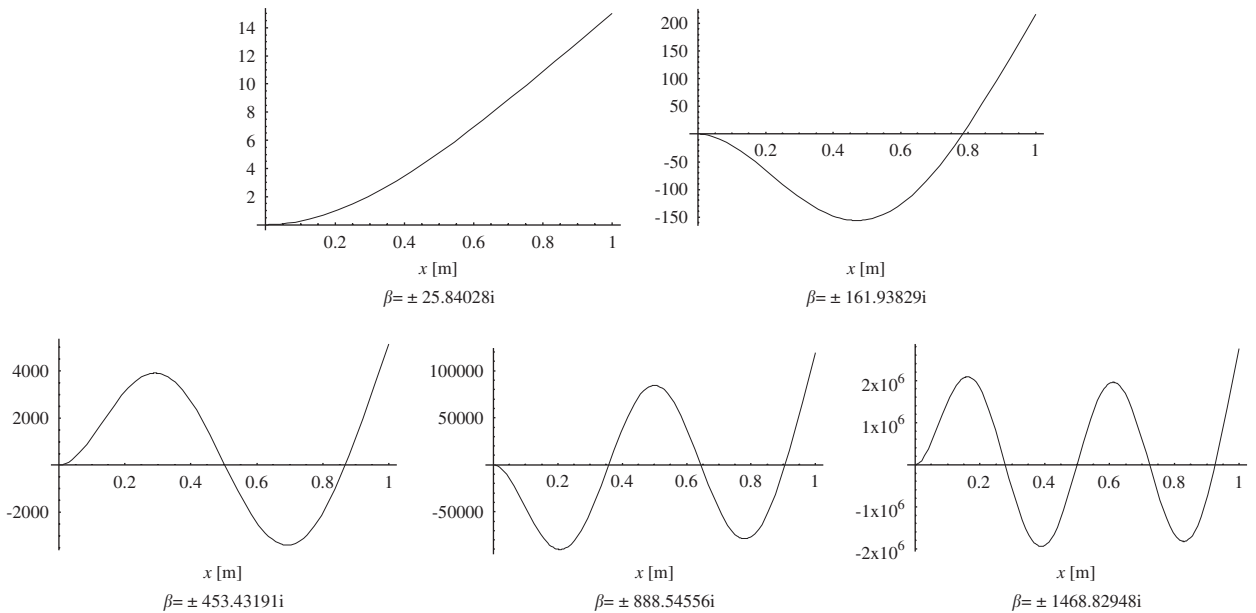


Fig. 2. Plots of eigenfunctions corresponding to the “first” five eigenvalues, for the external viscous damping coefficient $\sigma = 0$ and the viscous damping coefficient $d = 0$ at the free end.

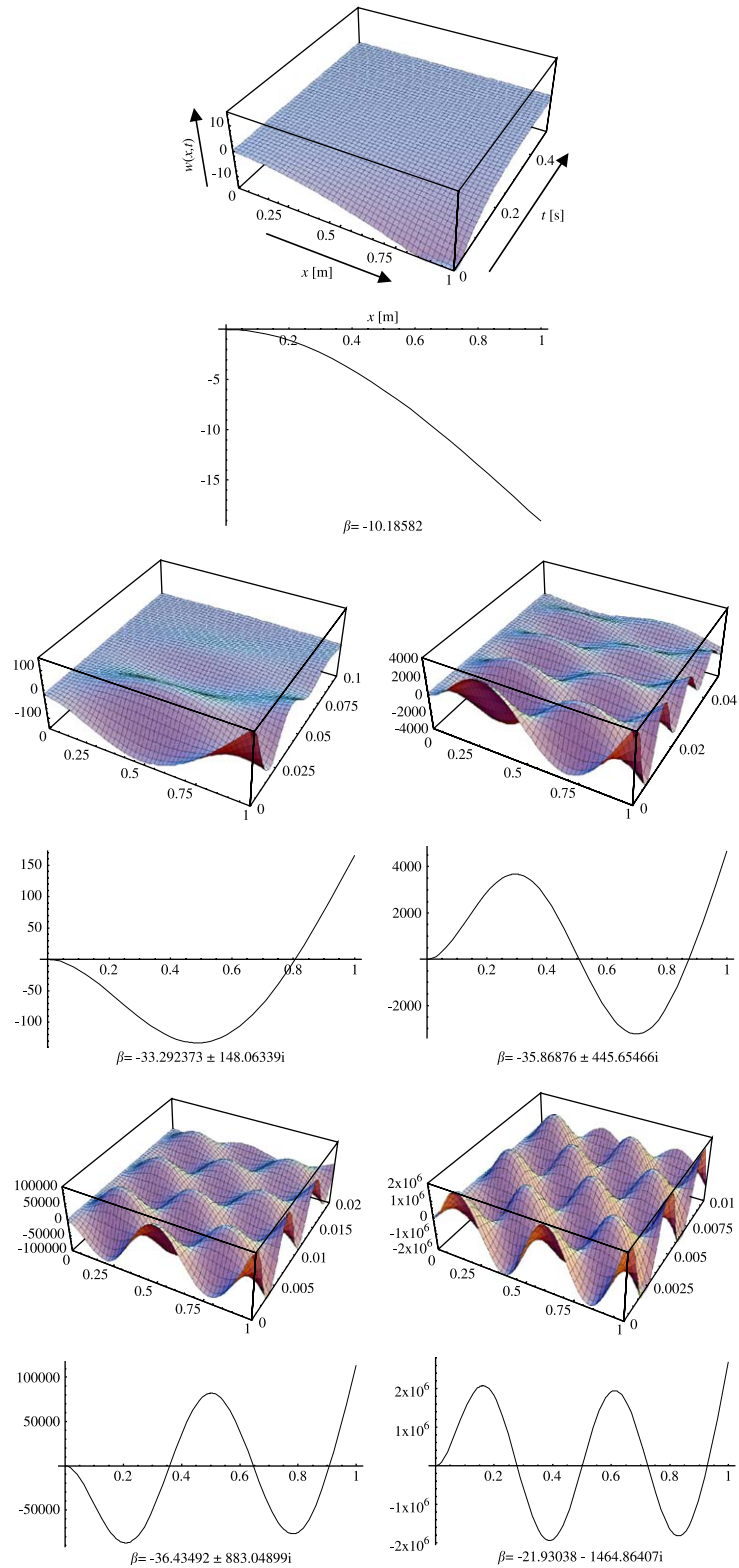


Fig. 3. Three-dimensional plots of $w_i(x,t)$ and eigenfunctions corresponding to the “first” five eigenvalues, for the external viscous damping coefficient $\sigma = 10 \text{ kg/ms}$ and the viscous damping coefficient $d = 10 \text{ kg/s}$ at the free end.

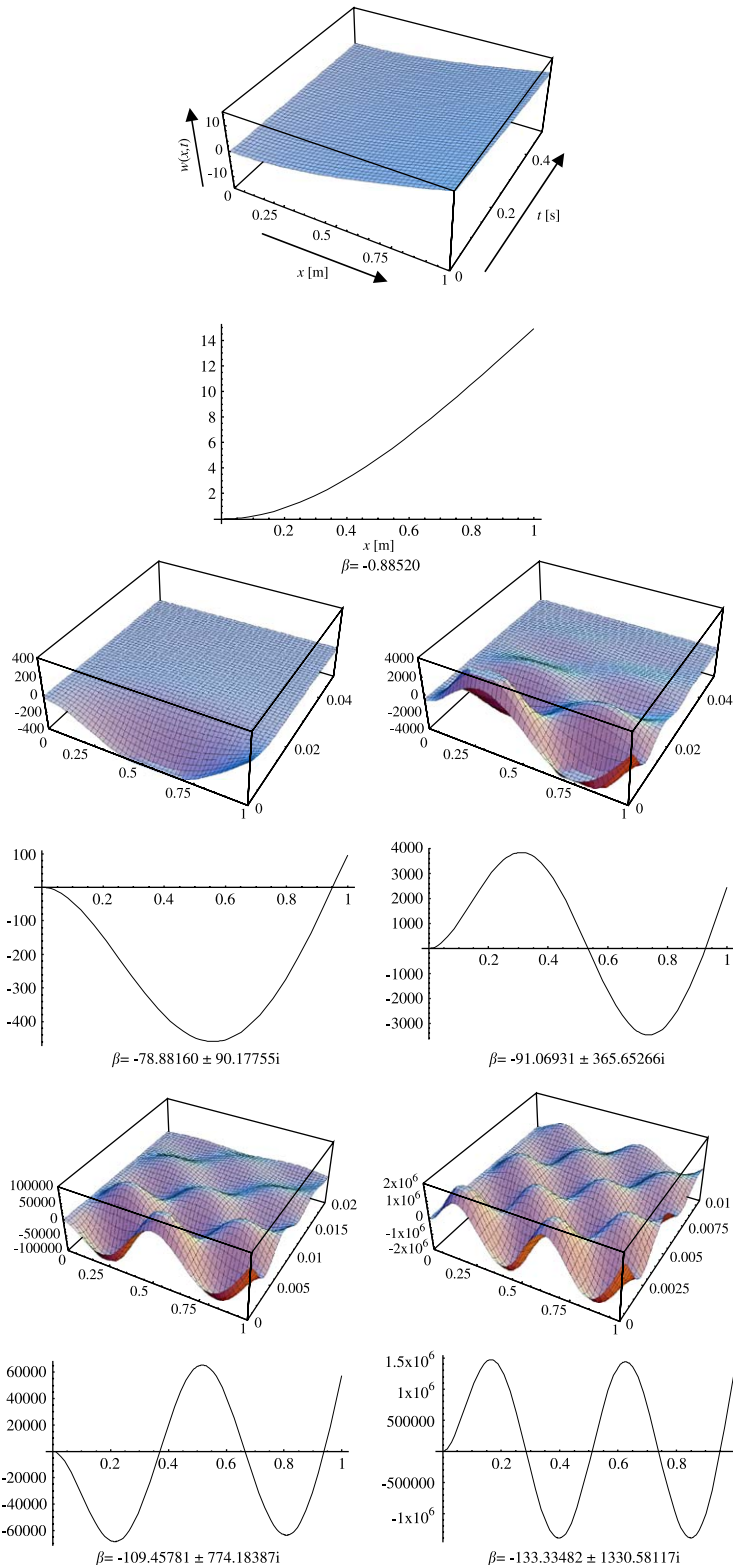


Fig. 4. Three-dimensional plots of $w_i(x,t)$ and eigenfunctions corresponding to the “first” five eigenvalues, for the external viscous damping coefficient $\sigma = 100$ kg/ms and the viscous damping coefficient $d = 100$ kg/s at the free end.

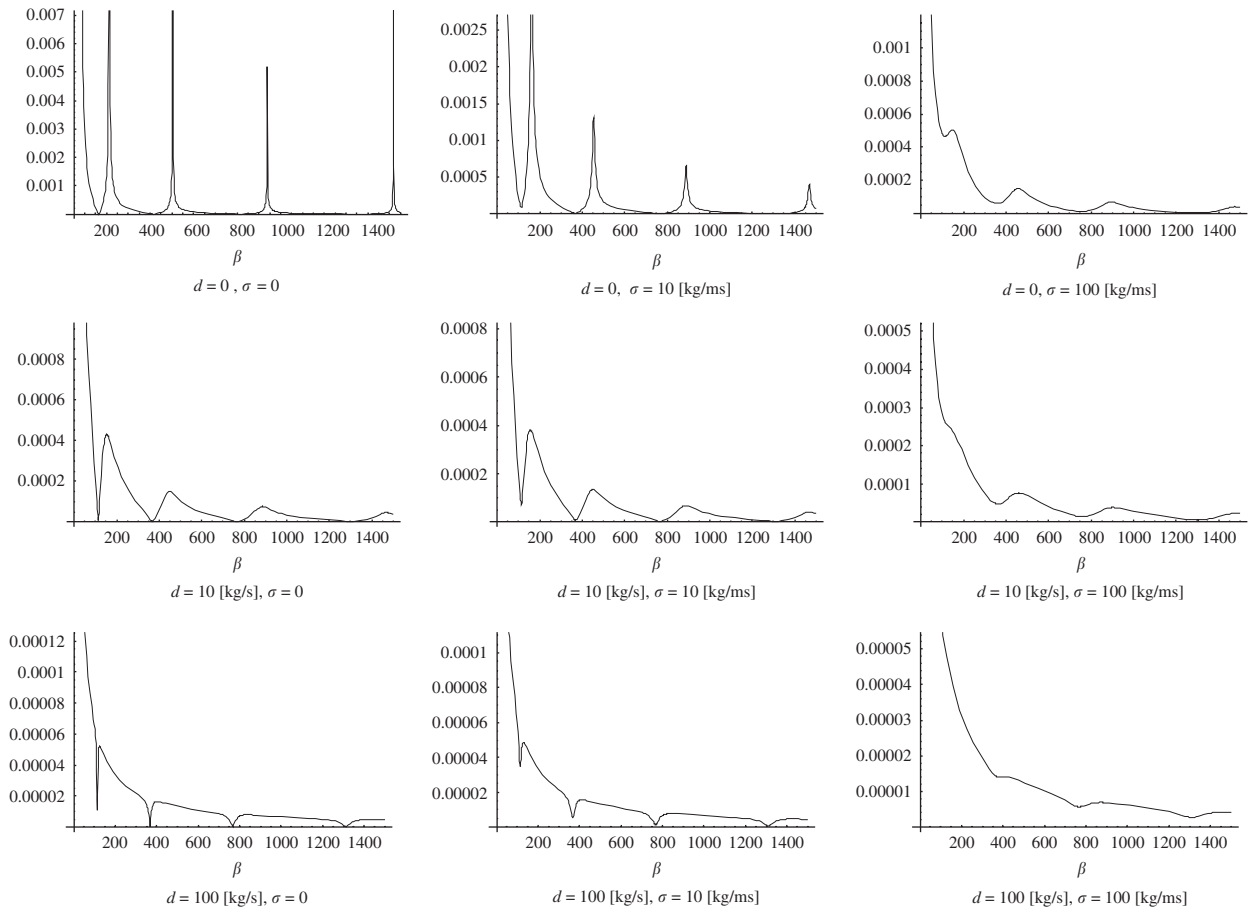


Fig. 5. Frequency responses of the beam for the three different viscous damping coefficients d at the free end and the three different external viscous damping coefficients σ .

Prior to these numerical evaluations, eigenvalues of the beam subjected to distributed external viscous damping in-span and with a viscous end condition by a single damper are determined. Table 1 gives the “first” five eigenvalues of the system for three different viscous damping coefficients d at the free end and for three different external viscous damping coefficients σ while the other parameters are kept constant. It is seen that the physical parameters in the damped case lead to both overdamped and underdamped “modes”. The numerical values in the first columns represent the results of finding the roots of the determinantal equation in Eq. (16), whereas those of the second columns are results of finding the roots of the determinant in Eq. (83) based on the solution through the boundary-value problem formulation.

Fig. 2 reflects the plots of eigenfunctions corresponding to the “first” five eigenvalues for the external viscous damping coefficient $\sigma = 0$ and the viscous damping coefficient $d = 0$ at the free end.

Fig. 3 shows both the three-dimensional plots of $w_i(x,t)$ and eigenfunctions corresponding to the “first” five eigenvalues for the external viscous damping coefficient $\sigma = 10$ (kg/ms) and the viscous damping coefficient $d = 10$ (kg/s) at the free end. The first pair of plots in Fig. 3, as the three-dimensional plots of $w_i(x,t)$ and eigenfunctions corresponding to the “first” eigenvalue, represents an overdamped mode, whereas the other modes are underdamped. It is seen from Fig. 3 that the amplitudes of the intersection curves corresponding to increasing t values decrease in accordance with the damped character of the system.

Fig. 4 represents both the three-dimensional plots of $w_i(x,t)$ and eigenfunctions corresponding to the “first” five eigenvalues for the external viscous damping coefficient $\sigma = 100$ (kg/ms) and the viscous damping coefficient $d = 100$ (kg/s) at the free end. The first pair of plots in Fig. 4, as the three-dimensional plots of $w_i(x,t)$ and eigenfunctions corresponding to the “first” eigenvalue, represents an overdamped mode, whereas

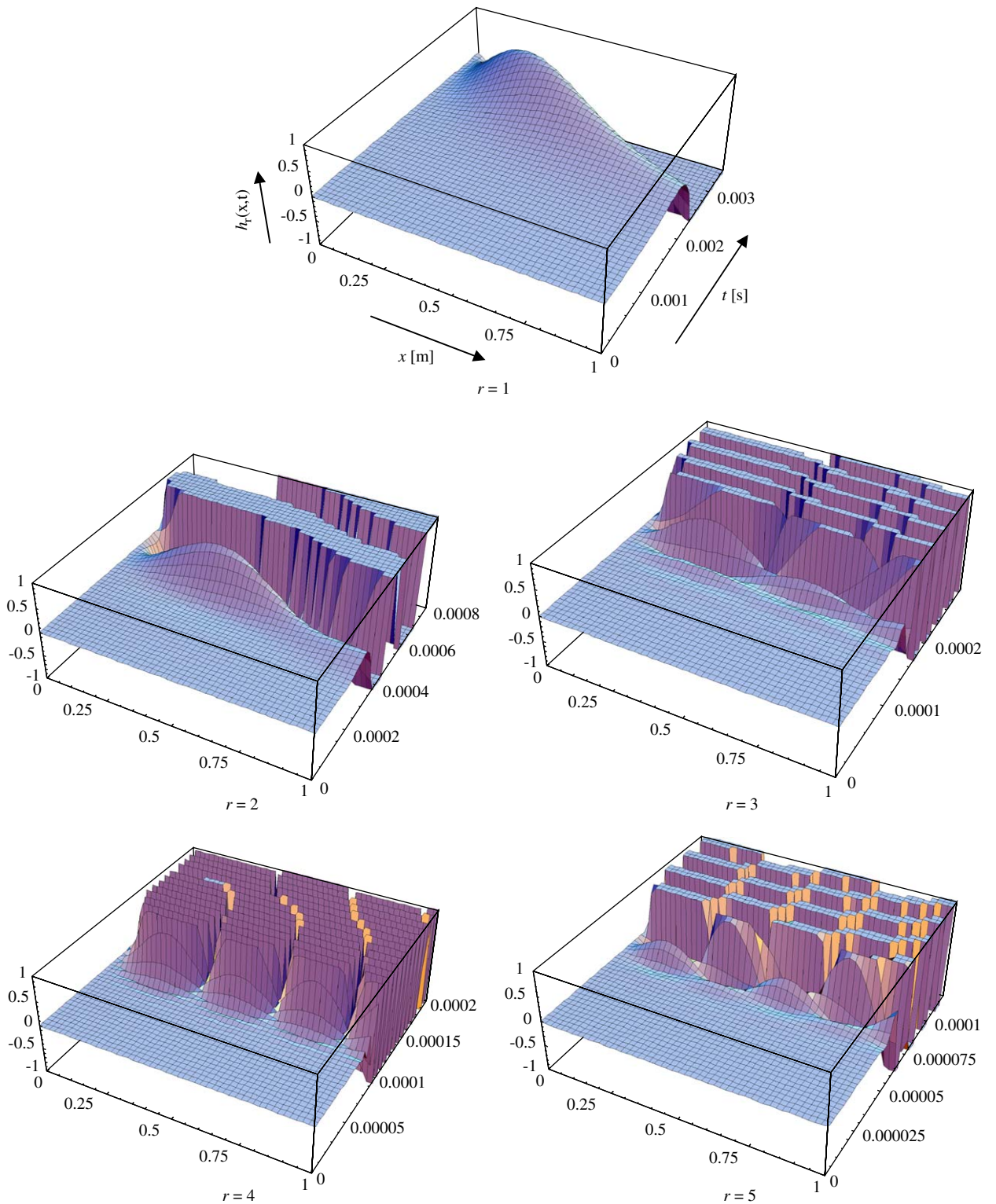


Fig. 6. Three-dimensional plots of the modal impulse response functions corresponding to the “first” five eigenvalues, for the external viscous damping coefficient $\sigma = 0$ and the viscous damping coefficient $d = 0$ at the free end.

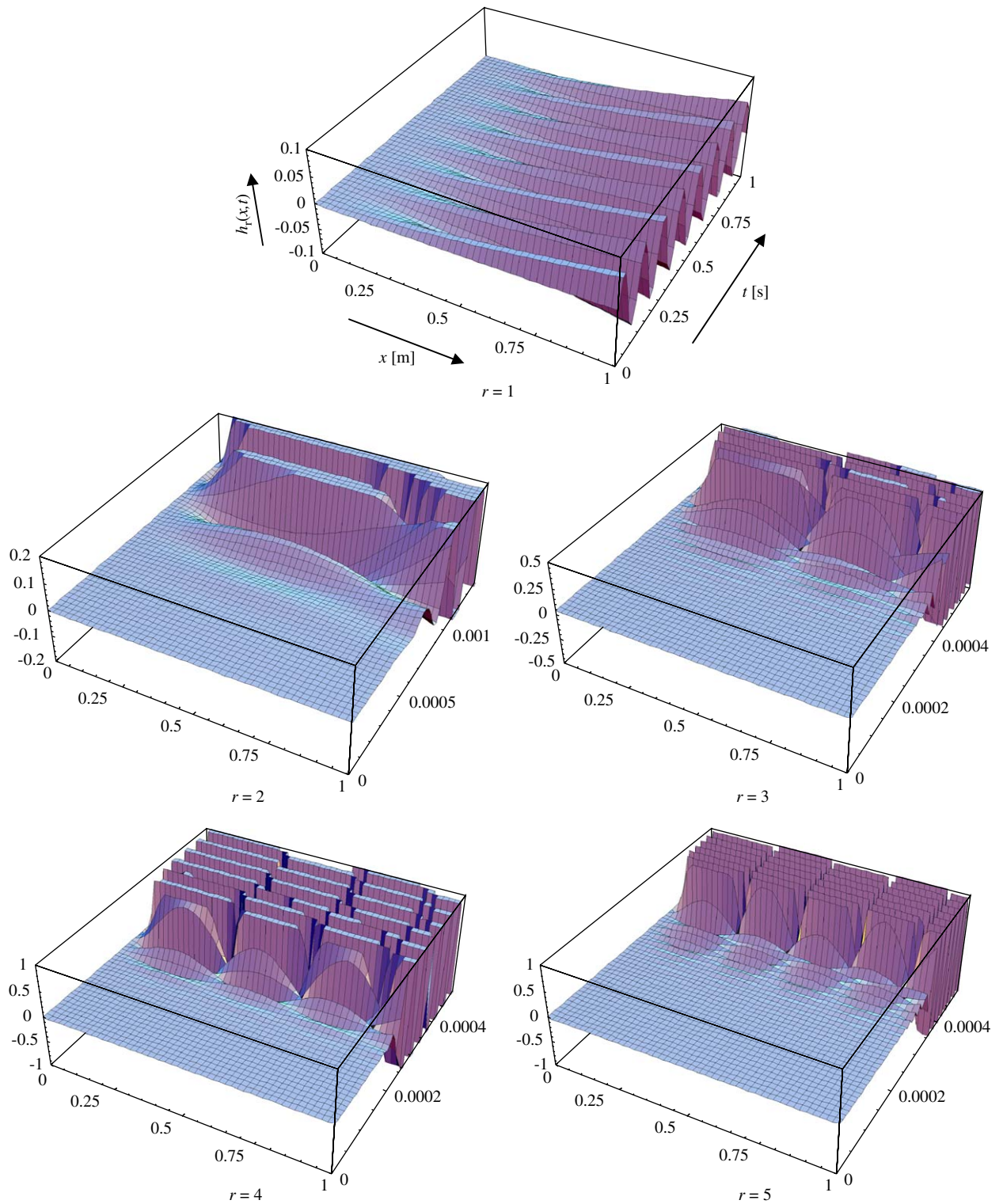


Fig. 7. Three-dimensional plots of the modal impulse response functions corresponding to the “first” five eigenvalues, for the external viscous damping coefficient $\sigma = 10$ kg/ms and the viscous damping coefficient $d = 10$ kg/s at the free end.

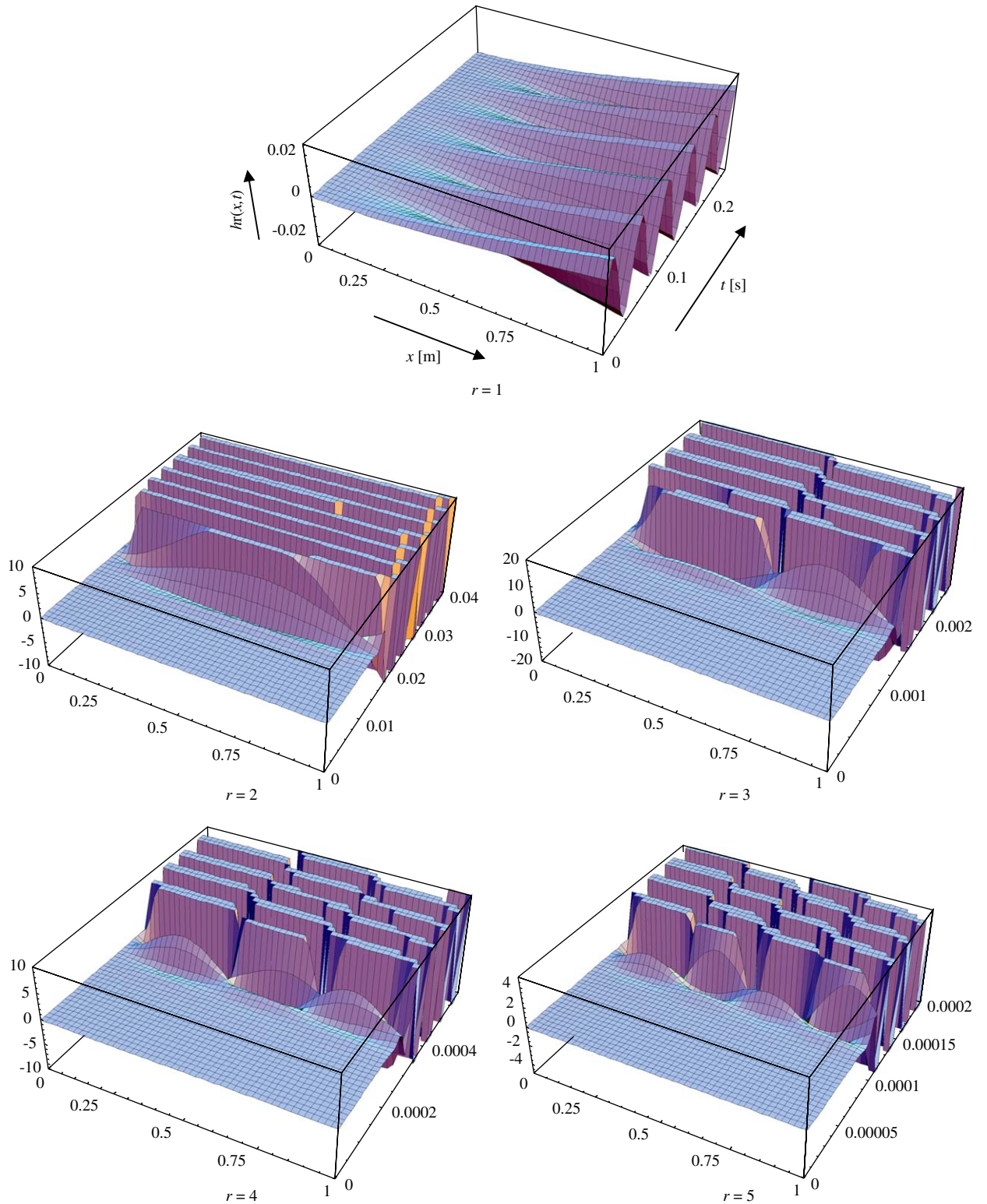


Fig. 8. Three-dimensional plots of the modal impulse response functions corresponding to the “first” five eigenvalues, for the external viscous damping coefficient $\sigma = 100$ kg/ms and the viscous damping coefficient $d = 100$ kg/s at the free end.

the other modes are underdamped. Fig. 4 illustrates that the amplitudes of the intersection curves corresponding to increasing t values decrease in accordance with the damped character of the system.

Fig. 5 gives the frequency responses of the beam for different viscous damping coefficients d at the free end and for different external viscous damping coefficients σ while the other parameters are kept constant. Fig. 5 is obtained from formula (62), i.e., the present theory, where $n = 15$ is taken in the series expansion (38), and β_1^*L to β_{15}^*L in Eq. (44) are taken from Ref. [6], which are correct up to 12 decimal places. This figure clarifies that increasing distributed external viscous damping in-span and the viscous damping coefficient at the boundary suppresses the frequency responses of the beam. Both of the distributed external viscous damping in-span and the viscous damping at the end are more effective at the resonance peaks in the system, as expected.

Fig. 6 reflects the three-dimensional plots of the modal impulse response functions, normalized to 1, corresponding to the “first” five eigenvalues for the external viscous damping coefficient $\sigma = 0$ and the viscous damping coefficient $d = 0$ at the free end. In this figure, for the undamped case, time history of the dynamic responses of the beam to an impulse are reported for the “first” five modes.

Similarly, Fig. 7 represents the three-dimensional plots of the modal impulse response functions corresponding to the “first” five eigenvalues for the external viscous damping coefficient $\sigma = 10$ (kg/ms) and the viscous damping coefficient $d = 10$ (kg/s) at the free end. The modal impulse response functions are again normalized for clarity of the modes.

Finally, Fig. 8 shows the three-dimensional plots of the modal impulse response functions corresponding to the “first” five eigenvalues for the external viscous damping coefficient $\sigma = 100$ (kg/ms) and the viscous damping coefficient $d = 100$ (kg/s) at the free end. For clarifying the modes, the modal impulse response functions are normalized too. Both distributed external viscous damping in-span and the single viscous damping at the boundary decrease magnitudes of the impulse response functions of the beam. However, it is seen that the single viscous damper at the free end is more effective on the suppression of the displacements of the free end.

4. Conclusions

This study is concerned with the dynamical analysis of a cantilevered Bernoulli–Euler beam subjected to distributed external viscous damping in-span with a viscous end condition by a single damper. In order to evaluate the vibration characteristics of the system, a procedure is presented where overdamped and underdamped modes are investigated. Here, both of them will be handled simultaneously, by setting to zero the determinant of the dynamic stiffness matrix of the system. The orthogonality conditions, which allow the decoupling of the equation of motion in terms of principal coordinates, are derived. Then, the complex frequency response function is obtained through a formula which was established for the receptance matrix, previously. Finally, the impulse response function of the system is evaluated for the dynamical analysis in time domain. Comparison of the numerical results obtained via a boundary value problem formulation justifies the approaches used here.

Appendix A

$$\begin{aligned}
 h_r^c(x, t) = & ((1 + i)e^{(-1-i)(2L+x)\beta_i + i\omega_i t}(-1 + ie^{2iL\beta_i} - ie^{2L\beta_i} + e^{(2+2i)L\beta_i})(e^{(L+ix)\beta_i} - ie^{((1+2i)(L+ix)\beta_i} \\
 & + (1 - i)e^{(2+i)L+ix)\beta_i} + e^{(iL+x)\beta_i} + (1 + i)e^{((1+2i)L+x)\beta_i} + ie^{((2+i)L+x)\beta_i} - ie^{(iL+(1+2i)x)\beta_i} \\
 & - (1 + i)e^{(L+(1+2i)x)\beta_i} - e^{((2+i)L+(1+2i)x)\beta_i} - (1 - i)e^{(iL+(2+i)x)\beta_i} + ie^{(L+(2+i)x)\beta_i} - e^{((1+2i)L+(2+i)x)\beta_i})\beta_i) \\
 & / (3(4 \sin((1 + i)\beta_i L) + (1 - i) \sin(2\beta_i L) + \sin((2 + 2i)\beta_i L) - 4 \sinh((1 + i)\beta_i L) \\
 & - (1 - i) \sinh(2\beta_i L) - \sinh((2 + 2i)\beta_i L)(\sigma + i2m\omega_i) + 2\beta_i((1 - i)\sigma L(-i2 \cos((1 + i)\beta_i L) \\
 & + \cos(2\beta_i L)) + d((-1 + i) \cos(\beta_i L) - i \cos((1 + 2i)\beta_i L) + \cos((2 + i)\beta_i L) \\
 & + (1 + i)\sigma L(2 \cosh((1 + i)\beta_i L) + i \cosh(2\beta_i L)) + d((-1 + i) \cosh(\beta_i L) - i \cosh((1 + 2i)\beta_i L) \\
 & + \cosh((2 + i)\beta_i L) + (1 - i) \sin(\beta_i L) + \sin((1 + 2i)\beta_i L) + \sin((2 + i)\beta_i L) - (1 - i) \sinh(\beta_i L) \\
 & - \sinh((1 + 2i)\beta_i L) - \sinh((2 + i)\beta_i L) - (4 + 4i)mL(\sin(\beta_i L) + \sinh(\beta_i L)^2\omega_i))
 \end{aligned}$$

Appendix B

$$\kappa = 16 EI(EI\beta^3 + \cosh(\beta L)(EI\beta^3 \cos(\beta L) + id\Omega \sin(\beta L) - id\Omega \cos(\beta L) \sinh(\beta L)),$$

$$\begin{aligned} \chi_1 = & e^{(-1-i)(1+\gamma)\beta L} EI(2e^{((1+i)+i\gamma)\beta L} EI\beta^3 + (1+i)e^{((1+i)+(1+2i)\gamma)\beta L} EI\beta^3 + e^{(1+2i)\gamma\beta L}((1+i)EI\beta^3 \\ & - 2d\Omega) + e^{(2+i\gamma)\beta L}(EI\beta^3 - (1+i)d\Omega) + e^{((2-2i)+i\gamma)\beta L}(EI\beta^3 + (1-i)d\Omega) \\ & + e^{(2+i)\gamma\beta L}(-iEI\beta^3 + (1+i)d\Omega) + ie^{(2i+(2+i)\gamma)\beta L}(EI\beta^3 + (1+i)d\Omega) + e^{(2+i)\gamma\beta L}((1-i)EI\beta^3 + 2d\Omega)) \end{aligned}$$

$$\begin{aligned} \chi_2 = & -e^{(-1-i)(1+\gamma)\beta L} EI((1+i)e^{((1+i)+\gamma)\beta L} EI\beta^3 + (1-i)e^{((1+i)+(1+2i)\gamma)\beta L} EI\beta^3 + 2e^{((1+i)+(2+i)\gamma)\beta L} EI\beta^3 \\ & + e^{(2+(1+2i)\gamma)\beta L}((1-i)EI\beta^3 - 2d\Omega) + e^{(2+i)\gamma\beta L}(EI\beta^3 - (1-i)d\Omega) + e^{(2+i)\gamma\beta L}(iEI\beta^3 + (1-i)d\Omega) \\ & - ie^{((2+2i)+i\gamma)\beta L}(EI\beta^3 + (1+i)d\Omega) + e^{(2i+(2+i)\gamma)\beta L}(EI\beta^3 + (1+i)d\Omega) \\ & + e^{((2+2i)+\gamma)\beta L}((1+i)EI\beta^3 + 2d\Omega)) \end{aligned}$$

$$\begin{aligned} \chi_3 = & e^{(-1-i)(1+\gamma)\beta L} EI((-1+i)e^{((1+i)+\gamma)\beta L} EI\beta^3 + 2ie^{((1+i)+\gamma)\beta L} EI\beta^3 + (1+i)e^{((1+i)+(2+i)\gamma)\beta L} EI\beta^3 \\ & + e^{(2+i)\gamma\beta L}((1+i)EI\beta^3 - 2d\Omega) + e^{(2+(1+2i)\gamma)\beta L}(EI\beta^3 - (1+i)d\Omega) - e^{(1+2i)\gamma\beta L}(EI\beta^3 - (1-i)d\Omega) \\ & + e^{((2+2i)+\gamma)\beta L}(iEI\beta^3 + (1+i)d\Omega) + ie^{(2i+\gamma)\beta L}(EI\beta^3 + (1+i)d\Omega) + e^{(2+i)\gamma\beta L}((-1+i)EI\beta^3 + 2d\Omega)), \end{aligned}$$

$$\begin{aligned} \chi_4 = & -e^{(-1-i)(1+\gamma)\beta L} EI((1+i)e^{((1+i)+i\gamma)\beta L} EI\beta^3 + 2ie^{((1+i)+(1+2i)\gamma)\beta L} EI\beta^3 - (1-i)e^{((1+i)+(2+i)\gamma)\beta L} EI\beta^3 \\ & + e^{(2i+(2+i)\gamma)\beta L}((-1+i)EI\beta^3 - 2d\Omega) + ie^{(1+2i)\gamma\beta L}(EI\beta^3 - (1-i)d\Omega) \\ & + e^{(2+(1+2i)\gamma)\beta L}(iEI\beta^3 + (1-i)d\Omega) - e^{((2+2i)+\gamma)\beta L}(EI\beta^3 + (1-i)d\Omega) \\ & + ie^{(2i+\gamma)\beta L}(EI\beta^3 + (1+i)d\Omega) + e^{((2+2i)+i\gamma)\beta L}((1+i)EI\beta^3 + 2d\Omega)), \end{aligned}$$

$$\begin{aligned} \chi_5 = & e^{(-1-i)(1+\gamma)\beta L} EI(-2e^{((1+i)+i\gamma)\beta L} EI\beta^3 + (1-i)e^{((1+i)+\gamma)\beta L} EI\beta^3 + (1+i)e^{((1+i)+(2+i)\gamma)\beta L} EI\beta^3 \\ & + e^{(1+2i)\gamma\beta L}((1+i)EI\beta^3 - 2d\Omega) - e^{i\gamma\beta L}(EI\beta^3 - (1-i)d\Omega) + e^{(2+i)\gamma\beta L}(-iEI\beta^3 + (1+i)d\Omega) \\ & - e^{(2+\gamma)\beta L}(EI\beta^3 + (1+i)d\Omega) + ie^{(2i+(2+i)\gamma)\beta L}(EI\beta^3 + (1+i)d\Omega) + e^{(2+i)\gamma\beta L}((1-i)EI\beta^3 + 2d\Omega)), \end{aligned}$$

$$\begin{aligned} \chi_6 = & e^{-i(1+i)\gamma\beta L} EI((-1-i)e^{(i+\gamma)\beta L} EI\beta^3 - (1-i)e^{(i+(1+2i)\gamma)\beta L} EI\beta^3 + e^{((1+2i)+\gamma)\beta L}((-1-i)EI\beta^3 - 2d\Omega) \\ & + e^{(1+(2+i)\gamma)\beta L}(EI\beta^3 - (1+i)d\Omega) + e^{(1+i)\gamma\beta L}(-iEI\beta^3 - (1-i)d\Omega) + e^{((1+2i)+(2+i)\gamma)\beta L}(EI\beta^3 + (1-i)d\Omega) \\ & + e^{((1+2i)+i\gamma)\beta L}(iEI\beta^3 + (1+i)d\Omega) + e^{(1+(1+2i)\gamma)\beta L}((-1+i)EI\beta^3 + 2d\Omega)), \end{aligned}$$

$$\begin{aligned} \chi_7 = & e^{(-1-i)(1+\gamma)\beta L} EI((-1+i)e^{((1+i)+i\gamma)\beta L} EI\beta^3 - 2ie^{((1+i)+\gamma)\beta L} EI\beta^3 + (1+i)e^{((1+i)+(2+i)\gamma)\beta L} EI\beta^3 \\ & + e^{(2+i)\gamma\beta L}((1+i)EI\beta^3 - 2d\Omega) + e^{(2+(1+2i)\gamma)\beta L}(EI\beta^3 - (1+i)d\Omega) + e^{(2+\gamma)\beta L}(-iEI\beta^3 - (1-i)d\Omega) \\ & - e^{(1+2i)\gamma\beta L}(EI\beta^3 - (1-i)d\Omega) + e^{i\gamma\beta L}(-iEI\beta^3 + (1+i)d\Omega) + e^{(2+i)\gamma\beta L}((-1+i)EI\beta^3 + 2d\Omega)), \end{aligned}$$

$$\begin{aligned} \chi_8 = & -e^{-(1+(1+i)\gamma)\beta L} EI((1+i)e^{(1+i\gamma)\beta L} EI\beta^3 - 2ie^{(1+(1+2i)\gamma)\beta L} EI\beta^3 - (1-i)e^{(1+(2+i)\gamma)\beta L} EI\beta^3 \\ & + e^{(i+(2+i)\gamma)\beta L}((-1+i)EI\beta^3 - 2d\Omega) - e^{((2+i)+\gamma)\beta L}(EI\beta^3 - (1-i)d\Omega) - ie^{((2+i)+(1+2i)\gamma)\beta L}(EI\beta^3 + (1-i)d\Omega) \\ & + e^{(i+\gamma)\beta L}(EI\beta^3 + (1+i)d\Omega) - ie^{(i+(1+2i)\gamma)\beta L}(EI\beta^3 + (1+i)d\Omega) + e^{((2+i)+i\gamma)\beta L}((1+i)EI\beta^3 + 2d\Omega)). \end{aligned}$$

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